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The accuracy of selected approximations for the reflection function of a semi-infinite turbid medium

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Abstract
This paper is devoted to studies of the accuracy of approximate equations for the reflection function of semi-infinite turbid media. It is assumed that the probability of photon absorption β by a particle is low. We derive quadratic and cubic terms of the expansion of the reflection function with respect to $\sqrt{\beta}$. Our results show that the cubic term makes a great improvement in the accuracy in the region of values of the similarity parameter that are not small.

1. Introduction

The reflection function of a semi-infinite turbid medium plays a central role in the radiative transfer theory [1, 2]. It obeys the nonlinear integral equation [1]. Nowadays there is no problem to calculate this function numerically and codes are freely available via Internet [3].

This in no way diminishes the importance of analytical results. In particular, such results can be obtained in the case of small probabilities β of photon absorption by particles [4–9]. For instance, it follows that for the reflection function $R_\infty(\xi, \eta, \varphi)$ of a semi-infinite plane parallel turbid medium as $\beta \to 0$,

$$R_\infty(\xi, \eta, \varphi) = R_\infty^0(\xi, \eta, \varphi) - yK_0(\xi)K_0(\eta), \qquad (1)$$

where $R_\infty^0(\xi, \eta, \varphi) \equiv R_\infty(\xi, \eta, \varphi)$ at $\beta = 0$, ξ is the cosine of the incidence angle, η is the cosine of the observation angle, φ is the azimuth angle,

$$y = 4\sqrt{\frac{\beta}{3(1-g)}} \qquad (2)$$

is the similarity parameter and

$$g = \frac{1}{2}\int_0^\pi p(\theta)\sin\theta\cos\theta\,\mathrm{d}\theta \qquad (3)$$

is the asymmetry parameter, which describes the degree of elongation of the phase function $p(\theta)$ in the forward direction.

The function $K_0(\xi)$ describes the angular distribution of light escaping from a turbid semi-infinite plane-parallel nonabsorbing layer with sources of radiation, located at infinity [6]. This is a smooth function of the escape angle due to the randomization of photon directions during their propagation from infinity to the border of a semi-infinite nonabsorbing medium. Actually $K_0(\xi)$ practically does not depend on the angular distribution of light in a single scattering event $p(\theta)$ (at least for $\xi > 0.2$). It follows, approximately, that [5]

$$K_0(\xi) = \tfrac{3}{7}(1 + 2\xi). \qquad (4)$$

We see that equation (1) allows us to relate the reflection function of an absorbing turbid medium as $\beta \to 0$ with that for a nonabsorbing turbid layer.

This result is of a considerable importance due to the fact that the function $R_\infty^0(\xi, \eta, \varphi)$ depends only on the phase function of a scattering medium and parameters ξ, η, φ. It does not depend on β by definition. Moreover, for selected classes of turbid media and viewing geometries, the function $R_\infty^0(\xi, \eta, \varphi)$ practically does not depend on the specific form of the phase function as well [10, 11]. This allows us to make the substitution of this function by a simple universal function, which depends only on the geometry [11].

Note that equation (1) can be applied to the determination of the spectral dependence $y(\lambda)$, where λ is the wavelength. This is of importance for the spectroscopy of turbid media [10].

The task of this paper is twofold. First of all, we derive next two terms of the expansion, given by equation (1), which

are proportional to y^2 and y^3 correspondingly. For this we will use well-known expansions for a plane albedo of a light scattering medium [7, 8]. Secondly, equation (1) and new expansions are compared with radiative transfer calculations using the code described by Mishchenko *et al* .

2. Theory

Clearly, in the general case for arbitrary values of y, we have, instead of equation (1),

$$R_\infty(\xi,\eta,\varphi)=\sum_{n=0}^{\infty}a_n y^n, \tag{5}$$

where, comparing equations (1) and (5), we have: $a_0 = R^0_\infty(\xi,\eta,\varphi)$ and $a_1 = -K_0(\xi)K_0(\eta)$. Coefficients a_n at $n > 1$ are unknown.

Our task is to find values of a_2 and a_3. For this, we will use the well-known expansion for the integral

$$r(\xi)=\frac{1}{\pi}\int_0^{2\pi}\mathrm{d}\varphi\int_0^1\eta\,\mathrm{d}\eta R_\infty(\xi,\eta,\varphi), \tag{6}$$

which is called the plane albedo. It gives us the total reflectance of a turbid layer at a given light incidence angle $\vartheta_0 = \arccos\xi$. Namely, it follows at small y that [7, 8]

$$r(\xi)=\sum_{n=0}^{3}b_n y^n, \tag{7}$$

where

$$\begin{aligned} &b_0=1, \qquad b_1=-K_0(\xi),\\ &b_2=\tfrac{3}{8}\left(pK_0(\xi)+\tfrac{3}{2}qV_0(\xi)\right),\\ &b_3=-\tfrac{3}{16}(u-vW_0(\xi))K_0(\xi) \end{aligned} \tag{8}$$

and

$$u=1-\tfrac{3}{2}g-\left(\varepsilon-\tfrac{11}{5}\right)q, \qquad v=(\varepsilon-h)q-\tfrac{3}{4}p^2,$$

$$\begin{aligned} &p=4\int_0^1 K_0(\xi)\xi^2\,\mathrm{d}\xi, \qquad q=\frac{1-g}{1-h},\\ &\varepsilon=6\int_0^1 K_0(\xi)\xi^3\,\mathrm{d}\xi,\\ &h=\frac{3}{4}\int_0^{\pi}p(\theta)\,\sin\theta\,\cos^2\theta\,\mathrm{d}\theta-\frac{1}{2}, \end{aligned} \tag{9}$$

$$\begin{aligned} &V_0(\xi)=\xi^2-\frac{1}{\pi}\int_0^{2\pi}\mathrm{d}\varphi\int_0^1 R^0_\infty(\xi,\eta,\varphi)\eta^3\,\mathrm{d}\eta,\\ &W_0(\xi)=\frac{K_2(\xi)}{2K_0(\xi)\int_0^1 K_2(\eta)\eta\,\mathrm{d}\eta}. \end{aligned}$$

Here $K_2(\xi)$ is the coefficient in the expansion of the escape function, given by

$$K(\xi)=\sum_{n=0}^{\infty}K_n(\xi)y^n. \tag{10}$$

It follows that [8]

$$K_1(\xi)=-\tfrac{3}{8}pK_0(\xi), \tag{11}$$

where $K_0(\xi)$ at $\xi > 0.2$ is given by equation (4).

The angular dependence $K_2(\xi)$, unlike the case of $K_1(\xi)$, differs from $K_0(\xi)$. One can find functions $V_0(\xi)$ and $W_0(\xi)$, which define $K_2(\xi)$ (see equations (9)) in figures 1 and 2. Symbols represent the data of Minin [8] for cloudy media and solid lines correspond to our approximate equations, obtained by fitting Minin's [8] numerical data. They are

$$\begin{aligned} &V_0(\xi)=\tfrac{4}{25}[8\xi^2-3\xi-2],\\ &W_0(\xi)=\tfrac{1}{3}+\tfrac{1}{4}\xi+\tfrac{205}{234}\xi^2. \end{aligned} \tag{12}$$

Importantly, equations (12) can be applied to media other than clouds. This is possible due to their low sensitivity to the detailed microstructure of a turbid medium in question [8]. Integrals for p and ε can be found analytically, using equation (4), which is also not sensitive to the microstructure of a disperse medium. The simple integration gives us

$$p=\tfrac{10}{7}, \qquad \varepsilon=\tfrac{117}{70}. \tag{13}$$

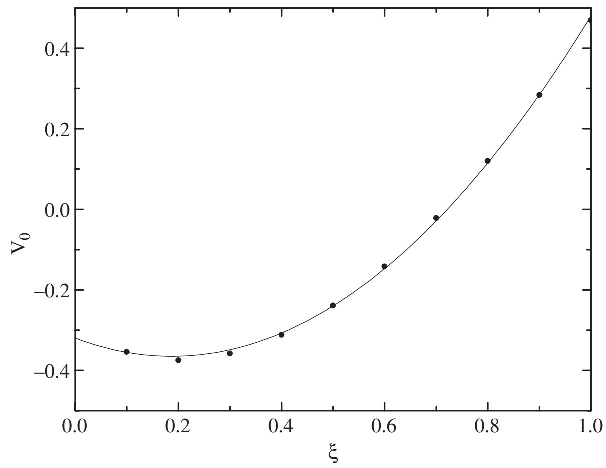

Figure 1. Function $V_0(\xi)$ calculated with equation (12) as compared to the numerical data of Minin [8] for cloudy media.

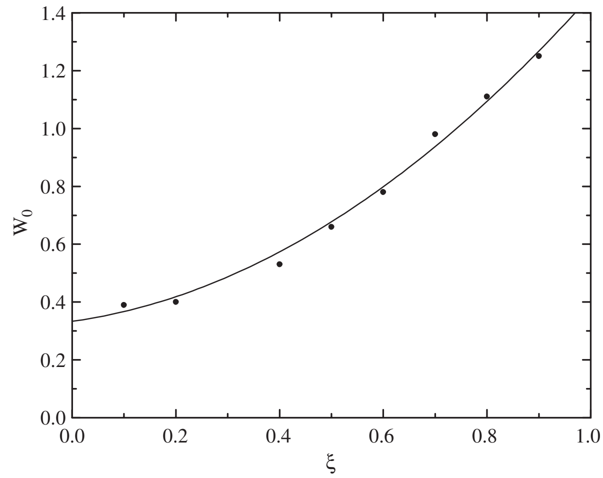

Figure 2. Function $W_0(\xi)$, calculated with equation (12) as compared to the numerical data of [8] for cloudy media.

Thus, we have (see equations (9))

$$u = 1 + \tfrac{37}{70}q - \tfrac{3}{2}g, \qquad v = (\tfrac{117}{70} - h)q - \tfrac{75}{49}. \tag{14}$$

We see that parameters u and v in equation (8) depend solely on values of g, h and the ratio $q = (1-g)/(1-h)$, which are determined by the phase function of a scattering medium in question.

In particular, it follows that for characteristic sizes of water droplets in terrestrial clouds: $g = 0.84$–0.87, $h = 0.76$–0.80, $q = 0.65$–0.67. These results were obtained using Mie calculations for the Υ water droplet size distribution $\Upsilon a^6 \exp(-9a/a_{\mathrm{ef}})$, where Υ is the normalization constant and a_{ef} is the effective radius, defined as the ratio of the third to the second moments of the droplet size distribution. The droplet effective radius a_{ef} was changed in the range 4–20 μm, which is characteristic for water clouds. The wavelength λ was equal to 0.65μm.

One can see that parameters g, h and q are not particularly sensitive to the microstructure of a cloudy medium. So for rapid estimations, one can use fixed values of g, h and q. For instance, using values of $g = 0.84$, $h = 0.76$ and $q = 2/3$, one obtains, for b_2 and b_3 in equation (7),

$$b_2 = \alpha K_0(\xi) + \gamma V_0(\xi), \qquad b_3 = -(\bar{\alpha} + \bar{\gamma} W_0(\xi))K_0(\xi). \tag{15}$$

Here we have (see equations (8), (13) and (14))

$$\alpha \equiv \tfrac{3}{8}p \approx 0.54, \qquad \gamma \equiv \tfrac{9}{16}q \approx 0.38,$$
$$\bar{\alpha} \equiv \tfrac{3}{16}u \approx 0.02, \qquad \bar{\gamma} \equiv -\tfrac{3}{16}v \approx 0.18$$

for water clouds. The values of α, γ, $\bar{\alpha}$ and $\bar{\gamma}$ for other types of disperse media can be easily found if integrals g and h (see equations (3) and (9)) are known.

So instead of equation (7), it follows that

$$r(\xi) = 1 - yK_0(\xi) + y^2(\alpha K_0(\xi) + \gamma V_0(\xi)) - y^3(\bar{\alpha} + \bar{\gamma} W_0(\xi))K_0(\xi). \tag{16}$$

Let us generalize equation (16) and consider the case of the reflection function $R_\infty(\xi, \eta, \varphi)$ in the framework of the same approach. Our derivation is not mathematically rigorous, but rather physically based. Its accuracy, however, appeared to be high as compared to the numerical solution of the radiative transfer equation (see next section). For this we will use equations (6) and (16), the reciprocity principle ($R_\infty(\xi, \eta, \varphi) \equiv R_\infty(\eta, \xi, \varphi)$) and following exact integral relations [8]:

$$2\int_0^1 K_0(\xi) W_0(\xi) \xi \, \mathrm{d}\xi = 1, \tag{17}$$

$$2\int_0^1 K_0(\xi) \xi \, \mathrm{d}\xi = 1, \tag{18}$$

$$\int_0^1 V_0(\xi) \xi \, \mathrm{d}\xi = 0, \tag{19}$$

$$\frac{1}{\pi}\int_0^{2\pi} \mathrm{d}\varphi \int_0^1 \eta \, \mathrm{d}\eta R_\infty^0(\xi, \eta, \varphi) = 1. \tag{20}$$

Then instead of equation (16), we have

$$\begin{aligned} R_\infty(\xi, \eta, \varphi) = {} & R_\infty^0(\xi, \eta, \varphi) - yK_0(\xi)K_0(\eta) \\ & + y^2[\alpha K_0(\xi)K_0(\eta) + \gamma(V_0(\xi) + V_0(\eta))] \\ & - y^3[(\bar{\alpha} + \bar{\gamma} W_0(\xi)W_0(\eta))K_0(\xi)K_0(\eta)]. \end{aligned} \tag{21}$$

Equation (16) follows from equation (21) using the definition (6) and integral relations (17)–(20). Due to the reciprocity principle, we can have in equation (21) only combinations

$$K_0(\xi) + K_0(\eta), \qquad K_0(\xi)K_0(\eta),$$
$$V_0(\xi) + V_0(\eta), \qquad V_0(\xi)V_0(\eta),$$
$$W_0(\xi) + W_0(\eta), \qquad W_0(\xi)W_0(\eta).$$

Combinations $K_0(\xi) + K_0(\eta)$, $V_0(\xi)V_0(\eta)$ and $W_0(\xi) + W_0(\eta)$ do not enter equation (21) because they are not consistent with equations (6) and (16). Note that equation (21) allows us to find the spherical albedo

$$r_{\mathrm{s}} = \frac{2}{\pi}\int_0^\pi \mathrm{d}\varphi \int_0^1 \xi \, \mathrm{d}\xi \int_0^1 \eta \, \mathrm{d}\eta R_\infty(\xi, \eta, \varphi). \tag{22}$$

Namely, it follows that

$$r_{\mathrm{s}} = 1 - y + cy^2 - dy^3, \tag{23}$$

where $c = \alpha \approx 0.54$, $d = \bar{\alpha} + \bar{\gamma} \approx 0.15$ for clouds or without substitution for values of p, q, u, v for arbitrary disperse media:

$$c = \tfrac{3}{8}p, \qquad d = \tfrac{3}{16}(u - v). \tag{24}$$

Equations (23) and (24) were earlier derived by Yanovitskij [7]. This confirms our calculations. Note that equation (23) reminds us of the following well-known expansion:

$$\mathrm{e}^{-y} = 1 - y + A_1 y^2 - A_2 y^3 + \cdots, \tag{25}$$

where $A_1 = 1/2$, $A_2 = 1/6$. Indeed we have, approximately, $c \approx A_1$, $d \approx A_2$ and most importantly,

$$r_{\mathrm{s}} = \mathrm{e}^{-y}. \tag{26}$$

This approximate phenomenological equation is known for a long time [4]. One can see that this simple formula approximately accounts for up to the third term in the expansion

$$r_\infty = \sum_{n=0}^{\infty} N_n y^n.$$

As we will see, it can be used even beyond the range of vadility of the exact asymptotic result (23) (particularly, at $y \approx 1$). Note that equation (26) can be generalized to describe the reflection function $R_\infty(\xi, \eta, \varphi)$ [4, 9]:

$$R_\infty(\xi, \eta, \varphi) = R_\infty^0(\xi, \eta, \varphi) \exp(-u(\xi, \eta, \varphi)y), \tag{27}$$

where

$$u(\xi, \eta, \varphi) = \frac{K_0(\xi)K_0(\eta)}{R_\infty^0(\xi, \eta, \varphi)}. \tag{28}$$

3. Accuracy of approximations

Let us study now the accuracy of equations (1), (21), (23), (26) and (27). For this we will use the numerical solution of the Ambarzumian's nonlinear integral equation for the reflection function of a semi-infinite random medium [3]. This equation is valid for arbitrary values of the probability of photon absorption. The results of exact calculations together with data obtained from equations (23) and (26) are presented in figure 3 as functions of the similarity parameter y. One can find errors Δ of these equations in figure 4.

Approximate equations and maximal values of y and $\omega_0 = 1 - \beta$, which allow to use them with accuracy better than 5%, are presented in table 1. We find also that if the tolerable error is 10%, then maximal values of y are equal to 0.45, 0.65, 1.0 and 1.4, depending on the approximation used (see table 1).

Note that taking into account the cubic term allows us to increase the value of $y^{\max}$ by 2.5 times (see table 1) as compared to the linear approximation (1). This is of importance for many natural media, including terrestrial and venerian clouds. The best accuracy is, however, due to the simple exponential solution (26) [4].

The reflection function $R_\infty(1, 0.6, 0)$ is presented in figure 5. Again we see that the exponential solution (27) has larger range of applicability than equation (21). It is simpler in structure also.

It should be stressed that data in figures 3 and 5 were obtained for different values of a_{ef} and, correspondingly, for different phase functions. They perfectly lie on a smooth

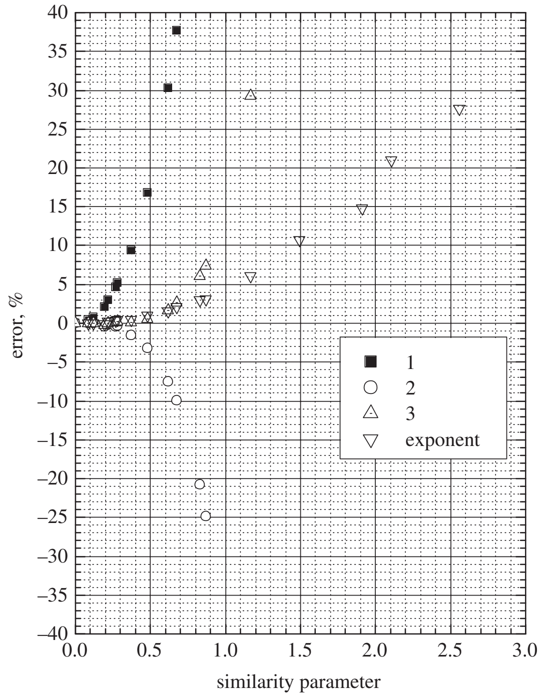

Figure 4. Errors, correspondent to data, in figure 3 (1—linear term, 2—linear and quadratic terms, 3—linear, quadratic and cubic terms, 4—exponent (26)).

Table 1. The maximal values of y and ω_0, which make the accuracy of equations better than 5%. The value of g was equal to 0.85.

R_∞	$y^{\max}$	$\omega_0^{\max}$
$1 - y$	0.3	0.9975
$1 - y + cy^2$	0.5	0.9930
$1 - y + cy^2 - dy^3$	0.75	0.9842
e^{-y}	1	0.9719

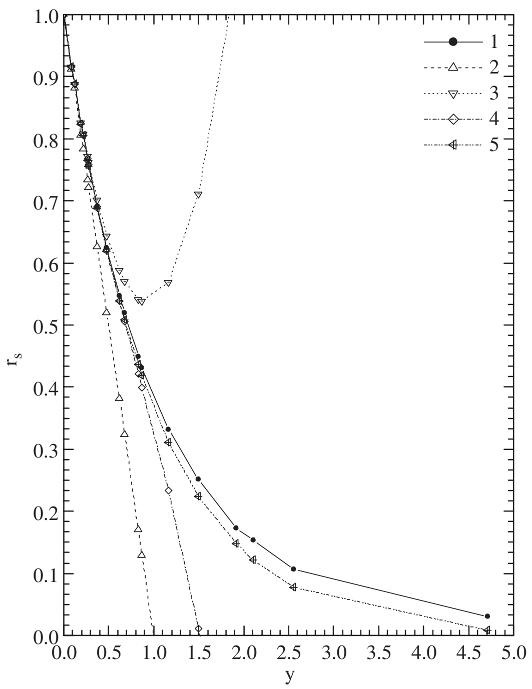

Figure 3. Dependence $r_s(y)$ according to equation (26) (line 1), (23) (line 2—the account only for the linear term, line 3—the account for the quadratic and linear terms, line 4—the account for the linear, quadratic and cubic terms) and exact radiative transfer calculations (5) for cloudy media at $\lambda = 1.549\,\mu$m, $a_{\mathrm{ef}} = 4$, 6 and 16 μm, and the refractive index of droplets $n = 1.3109 - i\chi$, where $\chi = 10^{-5}$, 10^{-4} and 10^{-3}.

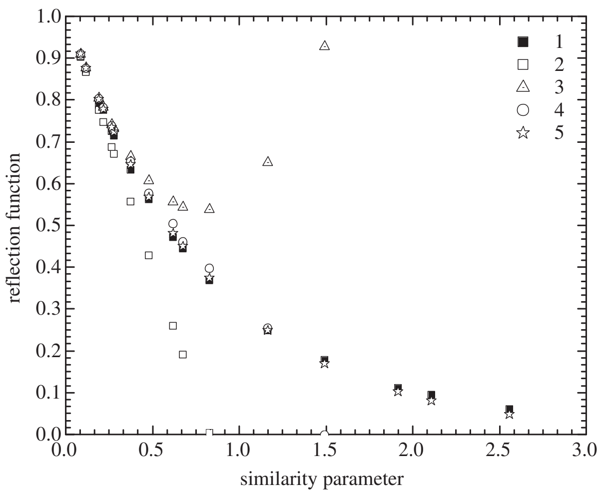

Figure 5. Dependence of the reflection function R_∞ (1, 0.6, 0) on the similarity parameter y, obtained from the exact radiative transfer calculations (1), equation (21) (2—the account only for the linear term, 3—the account for the linear and quadratic terms, 4—the account for the linear, quadratic and cubic terms) and equation (27) (stars) for cloudy media at $\lambda = 1.549\,\mu$m, $a_{\mathrm{ef}} = 4$, 6 and 16 μm, and the refractive index of droplets $n = 1.3109 - i\chi$, where $\chi = 10^{-5}$, 10^{-4} and 10^{-3}.

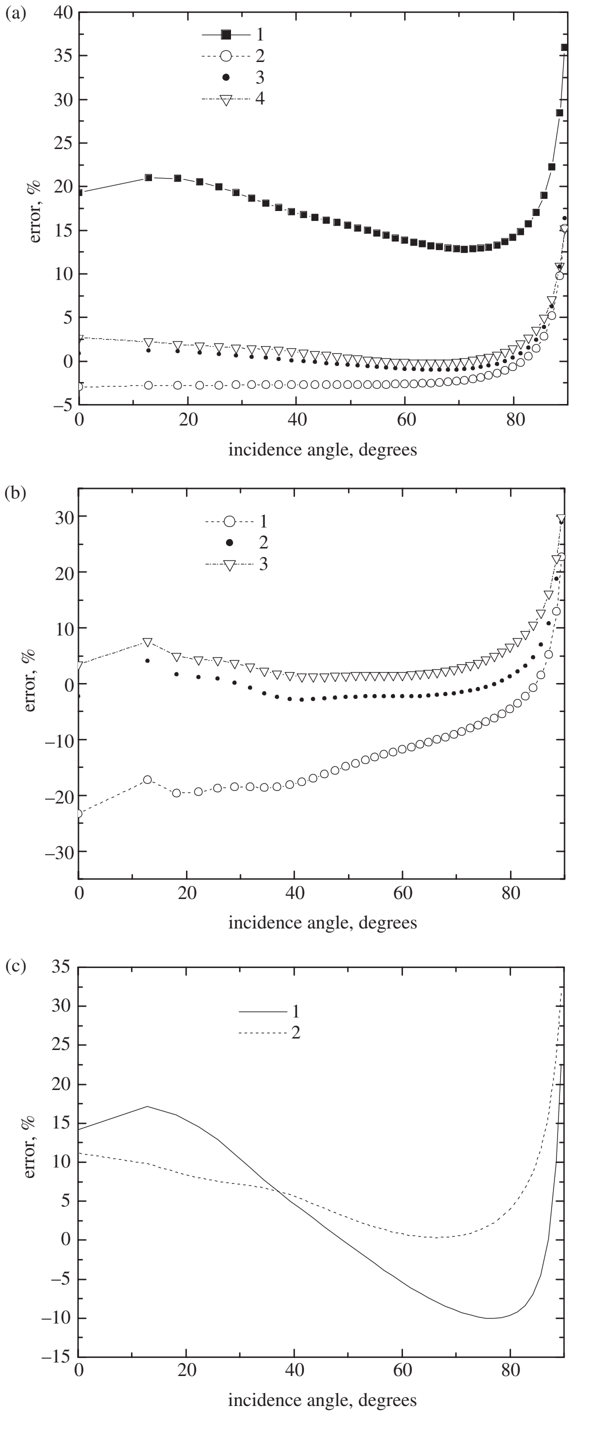

Figure 6. (*a*) Error of approximations, given by equation (21) (1—linear term, 2—linear and quadratic terms, 3—linear, quadratic and cubic terms) and equation (27) (triangles). The input parameters are the same as in figure 5, but $a_{ef} = 6\,\mu$m, $\chi = 10^{-4}$. (*b*) Same as in figure 6(*a*), but $a_{ef} = 4\,\mu$m, $\chi = 5 \times 10^{-4}$ (1—linear and quadratic terms, 2—linear, quadratic and cubic terms, 3—equation (27)). (*c*) Same as in figure 6(*a*), but $a_{ef} = 6\,\mu$m, $\chi = 5 \times 10^{-4}$ (1—linear, quadratic and cubic terms, 2—equation (27)).

curve as plotted against y. This suggests that media with different values of ω_0 and g but the same y have similar radiative characteristics. This is the numerical confirmation of the similarity principle [6].

Table 2. Values of ω_0, g and y for different a_{ef} and χ as obtained from the Mie theory at $\lambda = 1.549\,\mu$m, $n = 1.3109$.

a_{ef}	χ	ω_0	g	y
6	10^{-4}	0.9953	0.8210	0.37
4	5×10^{-4}	0.9850	0.7912	0.51
6	5×10^{-4}	0.9774	0.8253	0.83

The errors of approximations (21) and (27) for the reflection function at nadir observation and different angles of incidence are presented in figures 6(*a*)–(*c*). They were obtained at $\lambda = 1.549\,\mu$m, $a_{ef} = 6\,\mu$m, $n = 1.3109$, $\chi = 10^{-4}$ (figure 6(*a*)), $a_{ef} = 4\,\mu$m, $\chi = 5\times10^{-4}$ (figure 6(*b*)) and $a_{ef} = 6\,\mu$m, $\chi = 5 \times 10^{-4}$ (figure 6(*c*)). The values ω_0, g and y for all three cases were calculated with the Mie theory. They are presented in table 2.

We see that the error of the approximation depends on the incidence angle. It drastically increases for grazing light incidence. Note that equation (21) provides the best accuracy at small y. On the other hand, equation (27) has wider applicability range (see figure 6(*c*)) and can be applied even for relatively large values of y, where the total reflectance of light from a layer is low.

4. Conclusion

We have studied here the accuracy of different equations for the fundamental function of the radiative transfer theory, $R_\infty(\xi, \eta, \varphi)$. It was found that equation (21) is more accurate as $\beta \to 0$. However, the use of equations (26) and (27) is preferable if one needs to make rapid estimations or the interpretation of experimental data especially at $y \approx 1$.

Acknowledgments

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